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APPLICATION OF MODIFIED FRACTIONAL CALCULUS OPERATORS IN STATISTICAL DISTRIBUTIONS

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ABSTRACT

In this paper we investigate several properties of statistical distributions using Saigo's modified fractional integral operator. We here define a new family of distribution involving product combinations of elementary functions and hypergeometric functions, then apply it to certain generalized forms of univariate as well as multivariate statistical distributions. Further results giving the expectations, characteristic function, moment generating function, cummulative function of such special function distributions are also obtained.

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1. INTRODUCTION, DEFINITIONS AND PRELIMINARIES

Definition (1.1) In the theory of fractional calculus operators the modified Saigo Operator of fractional calculus [1] is defined by

$$D_{0,x;m}^{\alpha,\beta,\eta} f(x) = \frac{d}{dx^m}\left[\frac{x^{-m(\beta-\alpha)}}{\Gamma(1-\alpha)}\int_0^x (x^m - t^m)^{-\alpha} f(t).F(\beta-\alpha,1-\eta;1-\alpha;1-\frac{t^m}{x^m})d(t^m)\right]$$

Where $0 \le \alpha < 1$, $\beta,\eta \in \mathbb{R}$ and $m \in \mathbb{N}$

Definition (1.2) A fractional integral operator is defined by

(1.2.1) $$I_{0,x;m}^{\alpha,\beta,\eta} f(x)=\left[\frac{x^{-m(\alpha+\beta)}}{\Gamma(\alpha)}\int_0^x (x^m-t^m)^{\alpha-1} f(t).F(\alpha+\beta,-\eta,\alpha;1-\frac{t^m}{x^m})d(t^m)\right]$$

Where $\alpha,\beta,\eta\in\mathbb{C}$ with Re (α) >0 and $m\in\mathbb{N}$

$f(x)$ is any real continuous function defined on $(0,\infty)$ with the order

(1.2.2) $$f(x)=O\left(\left|x^{\varepsilon}\right|\right),\ x\to 0 \text{ for } \varepsilon>\max\{0,m(\beta-\eta)\}-m$$

And the multiplicity of $(x^m-t^m)^{\alpha-1}$ is removed by requiring that $\log(x^m-t^m)$ be real for $x>t$.

The kernel function $F={}_2F_1$ in (1.2.1) is well known gauss hypergeometric function [2].

Here we define a class $A(\beta,\eta)$ which satisfies the inequality for any ε

(1.2.3) $$\varepsilon>\max\{0,m(\beta-\eta)\}-m$$

Definition (1.3) The Gauss's hypergeometric function $F(a,b;c;x)$ in (1.2.1) is a special case of the generalized hypergeometric function defined by [2].

(1.3.1) $${}_pF_q\left[\begin{matrix}a_1,\dots,a_p;\\ b_1,\dots,b_q;\end{matrix}x\right]={}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;x)=\sum_{n=0}^{\infty}\frac{(a_1)_n\dots(a_p)_n}{(b_1)_n\dots(b_q)_n}\frac{x^n}{n!}$$

Definition (1.4) In our sequel, we shall use the generalized kampe de feriet function [9, p. 38] defined by

(1.4.1) $$F_{q:q_1,\dots,q_n}^{p:p_1,\dots,p_n}\left[\begin{matrix}(a_p):(b_{p_1}^1);\dots;(b_{p_n}^n);\\ (\alpha_q):(\beta_{q_1}^1);\dots;(\beta_{q_n}^n);\end{matrix}x_1,x_2,\dots,x_n\right]=\sum_{s_1,\dots,s_n=0}^{\infty}\phi(s_1,\dots,s_n)\frac{x_1^{s_1}}{s_1!}\dots\frac{x_n^{s_n}}{s_n!}$$

Where

(1.4.2) $$\phi(s_1,\dots,s_n)=\frac{\prod_{j=1}^{p}(a_j)_{s_1+\dots+s_n}\prod_{j=1}^{p_1}(b_j^1)_{s_1}\cdots\prod_{j=1}^{p_n}(b_j^n)_{s_n}}{\prod_{j=1}^{q}(\alpha_j)_{s_1+\dots+s_n}\prod_{j=1}^{q_1}(\beta_j^1)_{s_1}\cdots\prod_{j=1}^{q_n}(\beta_j^n)_{s_n}}$$

Definition (1.5) Expectation of Function

For any function g (x), the expectation of g(x) w.r.t. the p.d.f. $f(x)$ is defined by

(1.5.1) $$E\{g(x)\}=\int_{-\infty}^{\infty}f(x)g(x)dx$$

Definition (1.6) Characteristic Function

The characteristic function of a random variable x for the continuous probability distribution with density function $f(x)$ is defined as

(1.6.1) $$\phi_x(t) = E(e^{itx}) = \int_{-\infty}^{\infty} e^{itx} f(x)dx, i = \sqrt{-1}$$ and

t is a real parameter. $\phi_x(t)$ or $\phi(t)$ has the following properties

a) $\phi(t)$ is continuous in t

b) $\phi(t)$ is defined in every finite t interval

c) $\phi(0)=1$

d) $\phi(t)$ and $\phi(-t)$ are conjugate quantities

e) $|\phi(t)| \leq \int_{-\infty}^{\infty} |e^{itx}| f(x)dx = \int_{-\infty}^{\infty} f(x)dx = 1$

since $|\phi(t)| \leq 1$, characteristic function $\phi(t)$ always exists.

Definition (1.7) Moment Generating Function

The moment generating function of a continuous random variable x, about origin, having the probability function $f(x)$ is given by

(1.7.1) $$\underset{x=0}{M}(t) = E(e^{tx}) = \int e^{tx} f(x)dx$$

where t is the real parameter.

The m.g.f. about a point u is given by

(1.7.2) $$\underset{x=u}{M}(t) = E[e^{t(x-u)}] = \sum_i e^{t(x_i-u)}.p_i = \int e^{t(x-u)} f(x-u)dx$$

Definition (1.8) Cummulative Function

The cummulative probability function for the distribution function $F(t)$ is defined by

(1.8.1) $$F(t) = \int_{-\infty}^{t} f(x)dx$$

Definition (1.9) Multivariate Distribution

This is an extended form of the Dirichlet distribution [3, p.22]. The liouville's extension of the Dirichlet's result for any integrable function $\phi(x^m)$ is

(1.9.1) $$\int_{\Omega}\cdots\int (x_1^m)^{p_1-1}\cdots(x_n^m)^{p_n-1}\phi(x_1^m+\cdots+x_n^m)d(x_1^m)\cdots d(x_n^m) = B(p_1,\ldots,p_n)\int_{\mu^m}^{\nu^m} (x^m)^{\sum(p_i-1)}\phi(x^m)d(x^m)$$

Where the integrated region Ω is

(1.9.2) $$\Omega : \mu^m \leq x_1^m + \ldots + x_n^m \leq \rho^m (x_i^m \geq 0; i \in \{1,\ldots,n\}), 0 < \mu^m < \rho^m$$

and $m \in \mathbb{N}$, B means the generalized beta function

(1.9.3) $$B(p_1,\ldots,p_n) = \frac{\Gamma(p_1)\ldots\Gamma(p_n)}{\Gamma(p_1+\ldots+p_n)}$$

$$\left[\operatorname{Re}(p_i) > 0, i \in \{1,\ldots,n\}\right]$$

2. Results Required

In order to establish our main results, we require the following results:

Lemma 1: If $\lambda - 1 \in A(\beta,\eta), p_i \leq q_i and (p_i = q_i + 1, \max\left|r_i x^m\right| < 1), \forall i \in \{1,\ldots,n\}$ and $m \in \mathbb{N}$ then,

(2.1) $$I_{0,x:m}^{\alpha,\beta,\eta}\left[(x^m)^{\lambda-1}\prod_{i=1}^{n}\left\{ {}_{p_i}F_{q_i}\left[\begin{matrix}(a_{p_i}^i); \\ (b_{q_i}^i);\end{matrix} r_i x^m\right]\right\}\right] = \frac{\Gamma(\lambda)\Gamma(\lambda-\beta+\eta)}{\Gamma(\lambda-\beta)\Gamma(\lambda+\alpha+\eta)}(x^m)^{\lambda-\beta-1}$$

$$F_{2:q_1,\ldots,q_n}^{2:p_1,\ldots,p_n}\left[\begin{matrix}\lambda, \lambda-\beta+\eta & :(a_{p_1}^1);\ldots;(a_{p_n}^n); \\ \lambda-\beta, \lambda+\alpha+\eta & :(b_{q_1}^1);\ldots;(b_{q_n}^n);\end{matrix} r_1 x^m,\ldots,r_n x^m\right]$$

Proof: From [6] we have the result if $\lambda - 1 \in A(\beta,\eta), p_i \leq q_i and (p_i = q_i + 1, \max\left|r_i x^m\right| < 1), \forall i \in \{1,\ldots,n\}$, then

(2.2) $$I_{0,x}^{\alpha,\beta,\eta}\left[(x)^{\lambda-1}\prod_{i=1}^{n}\left\{ {}_{p_i}F_{q_i}\left[\begin{matrix}(a_{p_i}^i); \\ (b_{q_i}^i);\end{matrix} r_i x\right]\right\}\right] = \frac{\Gamma(\lambda)\Gamma(\lambda-\beta+\eta)}{\Gamma(\lambda-\beta)\Gamma(\lambda+\alpha+\eta)}(x)^{\lambda-\beta-1}.$$

$$F_{2:q_1,\ldots,q_n}^{2:p_1,\ldots,p_n}\left[\begin{matrix}\lambda, \lambda-\beta+\eta & :(a_{p_1}^1);\ldots;(a_{p_n}^n); \\ \lambda-\beta, \lambda+\alpha+\eta & :(b_{q_1}^1);\ldots;(b_{q_n}^n);\end{matrix} r_1 x,\ldots,r_n x\right]$$

Taking $m=1$ in (2.1) the L.H.S. of (2.1) is same as L.H.S. of (2.2).
Expanding each hypergeometric function in L.H.S. of (2.2) using (1.3.1) and then applying the formula from [7, p. 16]

(2.3) $$I_{0,x}^{\alpha,\beta,\eta}\left[(x)^k\right] = \frac{\Gamma(1+k)\Gamma(1+k-\beta+\eta)}{\Gamma(1+k-\beta)\Gamma(1+k+\alpha+\eta)}(x)^{k-\beta}$$

And interpreting the n-series by means of the definition (1.4.1), we arrive at the desired result (2.1) for $m=1$.
Now taking $m=2$ in L.H.S. of (2.1) we use

(2.4) $${}_pF_q\left[\begin{matrix}a_1,\ldots,a_p; \\ b_1,\ldots,b_q;\end{matrix} x^2\right] = \sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n}{(b_1)_n\ldots(b_q)_n}\frac{(x^2)^n}{n!}$$

And then we apply

(2.5) $$I_{0,x;2}^{\alpha,\beta,\eta}\left[(x^2)^k\right]=\frac{\Gamma(1+k)\Gamma(1+k-\beta+\eta)}{\Gamma(1+k-\beta)\Gamma(1+k+\alpha+\eta)}(x^2)^{k-\beta}$$

Similarly for m we have

(2.6) $${}_pF_q\left[\begin{matrix}a_1,\dots,a_p;\\ b_1,\dots,b_q;\end{matrix}x^m\right]=\sum_{n=0}^{\infty}\frac{(a_1)_n\dots(a_p)_n}{(b_1)_n\dots(b_q)_n}\frac{(x^m)^n}{n!}\quad\text{and}$$

(2.7) $$I_{0,x;m}^{\alpha,\beta,\eta}\left[(x^m)^k\right]=\frac{\Gamma(1+k)\Gamma(1+k-\beta+\eta)}{\Gamma(1+k-\beta)\Gamma(1+k+\alpha+\eta)}(x^m)^{k-\beta}$$

Again expanding each hypergeometric function in L.H.S. of (2.2) using (1.3.1) and then applying (2.7) and (1.4.1) with power of x as m, we get the desired result Lemma 1.

Similarly we can establish the following results:

Lemma 2: If $\lambda-1\in A(\beta,\eta), p_i\le q_i and (p_i=q_i+1,\max|r_ix^m|<1), \forall i\in\{1,\dots,n\}, h,\mu\in\mathbb{R}_+$ and $m\in\mathbb{N}$ then

(2.8) $$I_{0,x;m}^{\alpha,\beta,\eta}\left[(x^m)^{\lambda-1}(x^m+h^m)^{-\mu}\prod_{i=1}^{n}\left\{{}_{p_i}F_{q_i}\left[\begin{matrix}(a_{p_i}^i);\\ (b_{q_i}^i);\end{matrix}r_ix^m\right]\right\}\right]=\frac{\Gamma(\lambda)\Gamma(\lambda-\beta+\eta)}{\Gamma(\lambda-\beta)\Gamma(\lambda+\alpha+\eta)}(x^m)^{\lambda-\beta-1}h^{-m\mu}.$$

$$F_{2:q_1,\dots,q_n;0}^{2:p_1,\dots,p_n;1}\left[\begin{matrix}\lambda,\lambda-\beta+\eta & :(a_{p_1}^1);\dots;(a_{p_n}^n);\mu; \\ \lambda-\beta,\lambda+\alpha+\eta & :(b_{q_1}^1);\dots;(b_{q_n}^n);\_;\end{matrix} r_1x^m,\dots,r_nx^m,\frac{-x^m}{h^m}\right]$$

Remark: In (2.8) taking $h=1$ and $\mu=0$, we get (2.1). Hence in this way Lemma1 can be obtained from Lemma2.

Lemma 3: If $\lambda-1\in A(\beta,\eta), p_i\le q_i and (p_i=q_i+1,\max|r_ix^m|<1), \forall i\in\{1,\dots,n\}$, $m\in\mathbb{N}$ and $\mu\in\mathbb{R}_+$, then

(2.9) $$I_{0,x;m}^{\alpha,\beta,\eta}\left[(x^m)^{\lambda-1}e^{\mu x^m}\prod_{i=1}^{n}\left\{{}_{p_i}F_{q_i}\left[\begin{matrix}(a_{p_i}^i);\\ (b_{q_i}^i);\end{matrix}r_ix^m\right]\right\}\right]=\frac{\Gamma(\lambda)\Gamma(\lambda-\beta+\eta)}{\Gamma(\lambda-\beta)\Gamma(\lambda+\alpha+\eta)}(x^m)^{\lambda-\beta-1}.$$

$$F_{2:q_1,\dots,q_n}^{2:p_1,\dots,p_n}\left[\begin{matrix}\lambda,\lambda-\beta+\eta & :(a_{p_1}^1);\dots;(a_{p_n}^n); \\ \lambda-\beta,\lambda+\alpha+\eta & :(b_{q_1}^1);\dots;(b_{q_n}^n);\end{matrix} r_1x^m,\dots,r_nx^m,\mu x^m\right]$$

Lemma 4: If $\lambda-1\in A(\beta,\eta), p_i\le q_i, A_j\le B_j and$
$\left(p_i=q_i+1, A_j=B_j+1,\max|r_ix^m|<1,\max|s_jx^m|<1\right), \forall i\in\{1,\dots,n\}, j\in\{1,\dots,n'\}, m\in\mathbb{N}$
then

(2.10) $$I_{0,x;m}^{\alpha,\beta,\eta}\left[(x^m)^{\lambda-1}\prod_{i=1}^{n}\left\{{}_{p_i}F_{q_i}\left[\begin{matrix}(a_{p_i}^i);\\ (b_{q_i}^i);\end{matrix}r_ix^m\right]\right\}\prod_{j=1}^{n'}\left\{{}_{A_j}F_{B_j}\left[\begin{matrix}(c_{A_j}^j);\\ (d_{B_j}^j);\end{matrix}s_jx^m\right]\right\}\right]=\frac{\Gamma(\lambda)\Gamma(\lambda-\beta+\eta)}{\Gamma(\lambda-\beta)\Gamma(\lambda+\alpha+\eta)}$$

$$\cdot (x^m)^{\lambda-\beta-1} F^{2:p_1,\ldots,p_n;A_1,\ldots,A_n}_{2:q_1,\ldots,q_n;B_1,\ldots,B_n}\left[\begin{matrix}\lambda, \lambda-\beta+\eta & :(a^1_{p_1});\ldots;(a^n_{p_n});(c^1_{A_1});\ldots;(c^n_{A_n}); \\ \lambda-\beta, \lambda+\alpha+\eta & :(b^1_{q_1});\ldots;(b^n_{q_n});(d^1_{B_1});\ldots;(d^n_{B_n});\end{matrix} r_1x^m,\ldots,r_nx^m,s_1x^m,\ldots,s_nx^m\right].$$

Remark: This Lemma 4 can be manipulated to yield the Lemma's 1, 2 and 3.

Remark: Before investigating our main results we specify here that the use of Gaussian hypergeometric function ${}_2F_1$ and the generalized hypergeometric function ${}_pF_q$ in statistical distributions were earlier studied by A.M. Mathai, R.K.Saxena,and P.L.Sethi. ([4], [5] and [8])

3. A Generalized finite distribution

To investigate our main results we define a family of distributions having p.d.f. of the form

(3.1)
$$f(z)=\Im^{-1}(z^m-h^m)^{p-1}(k^m-z^m)^{\alpha-1}\,F\left(\alpha+\beta,-\eta;\alpha;\frac{k^m-z^m}{k^m-h^m}\right).$$
$$\prod_{i=1}^{n}\left\{{}_{p_i}F_{q_i}\left[\begin{matrix}(a^i_{p_i}); \\ (b^i_{q_i});\end{matrix} r_i(z^m-h^m)\right]\right\} \text{ for } h^m<z^m<k^m \text{ ; h>0}$$
$= 0$, otherwise

Provided that Re (α)>0, $\beta,\eta\in\mathbb{C}$, $p-1\in A(\beta,\eta)$, $p_i\le q_i$ and $\left(p_i=q_i+1, \max\limits_{1\le i\le n}\left|r_i(z^m-h^m)\right|<1\right)$ $\forall i\in\{1,\ldots,n\}, m\in\mathbb{N}$ and $h\neq k$ where

(3.2)
$$\Im=\tfrac{\Gamma(\alpha)\Gamma(p)\Gamma(p-\beta+\eta)}{\Gamma(p-\beta)\Gamma(p+\alpha+\eta)}\,(k^m-h^m)^{\alpha+p-1}$$
$$F^{2:p_1,\ldots,p_n}_{2:q_1,\ldots,q_n}\left[\begin{matrix}p, p-\beta+\eta & :(a^1_{p_1});\ldots;(a^n_{p_n}); \\ p-\beta, p+\alpha+\eta & :(b^1_{q_1});\ldots;(b^n_{q_n});\end{matrix} r_1(k^m-h^m),\ldots,r_n(k^m-h^m)\right]$$

Theorem 1: If $f(z)$ defined by (3.1) represents a p.d.f. then

$$\int_{-\infty}^{\infty} f(z)d(z^m)=1$$

Proof:

(3.3)
$$\int_{-\infty}^{\infty} f(z)d(z^m)=\int_{h^m}^{k^m} f(z)d(z^m)$$
$$=\Im^{-1}\int_{h^m}^{k^m}(z^m-h^m)^{p-1}(k^m-z^m)^{\alpha-1}\,F\left(\alpha+\beta,-\eta;\alpha;\frac{k^m-z^m}{k^m-h^m}\right).$$

$$\prod_{i=1}^{n}\left\{{}_{p_i}F_{q_i}\left[\begin{matrix}(a^i_{p_i});\\(b^i_{q_i});\end{matrix}r_i(z^m-h^m)\right]\right\}d(z^m)$$

Where $\Im$ is defined by (3.2).

A simple change of variables puts (3.3) in the form of

$$\int_{h^m}^{k^m} f(z)d(z^m)=\Im^{-1}\Gamma(\alpha)x^{m(\alpha+\beta)}\,I_{0,x;m}^{\alpha,\beta,\eta}\left[(x^m)^{p-1}\prod_{i=1}^{n}\left\{{}_{p_i}F_{q_i}\left[\begin{matrix}(a^i_{p_i});\\(b^i_{q_i});\end{matrix}r_ix^m\right]\right\}\right] \tag{3.4}$$

Where $k^m-h^m=x^m\ \ (k\neq h)$

Applying Lemma1 and (3.2) we arrive at the value unity on the right hand side showing that (3.1) is a p.d.f.

Theorem 2: If $f(z)$ is a p.d.f. of the form (3.1) then for any function $g(z)$, the expectation of $g(z)$ w.r.t. the p.d.f. $f(z)$ is given by

$$E\{g(z)\}=\aleph^{-1}F^{2:p_1,\ldots,p_n:A_1,\ldots,A_{n'}}_{2:q_1,\ldots,q_n:B_1,\ldots,B_{n'}} \tag{3.5}$$

$$\left[\begin{matrix}p,p-\beta+\eta & :(a^1_{p_1});\ldots;(a^n_{p_n});(c^1_{A_1});\ldots;(c^n_{A_{n'}});\\ p-\beta,p+\alpha+\eta & :(b^1_{q_1});\ldots;(b^n_{q_n});(d^1_{B_1});\ldots;(d^n_{B_{n'}});\end{matrix}r_1(k^m-h^m),\ldots,r_n(k^m-h^m),s_1(k^m-h^m),\ldots,s_{n'}(k^m-h^m)\right]$$

Provided $(k\neq h)$, $\mathrm{Re}(\alpha)>0$, $\beta,\eta\in\mathbb{C}$, $p-1\in A(\beta,\eta)$, $p_i\leq q_i+1$

$A_j\leq B_j+1;\forall i\in\{1,\ldots,n\}, j\in\{1,\ldots,n'\}, m\in\mathbb{N}$

In case of equality $\max\limits_{1\leq i\leq n}\left\{r_i(k^m-h^m)\right\}<1$,

$\max\limits_{1\leq j\leq n'}\left\{s_j(k^m-h^m)\right\}<1, (k\neq h)$

Where

$$\aleph=F^{2:p_1,\ldots,p_n}_{2:q_1,\ldots,q_n}\left[\begin{matrix}p,p-\beta+\eta & :(a^1_{p_1});\ldots;(a^n_{p_n});\\ p-\beta,p+\alpha+\eta & :(b^1_{q_1});\ldots;(b^n_{q_n});\end{matrix}r_1(k^m-h^m),\ldots,r_n(k^m-h^m)\right] \tag{3.6}$$

Proof: We consider a function $g(z)$ in terms of products of n' generalized hypergeometric functions given by

$$g(z)=\prod_{j=1}^{n'}\left\{{}_{A_j}F_{B_j}\left[\begin{matrix}(c^j_{A_j});\\(d^j_{B_j});\end{matrix}s_j(z^m-h^m)\right]\right\}$$

With $\max\limits_{1\leq j\leq n'}\left|s_j(k^m-h^m)\right|<1$

Let a p.d.f. $f(z)$ be defined by (3.1), then

$$E\{g(z)\}=\int_{-\infty}^{\infty}f(z)g(z)d(z^m) \tag{3.7}$$

$$=\int_{h^m}^{k^m}\Im^{-1}(z^m-h^m)^{p-1}(k^m-z^m)^{\alpha-1}F\left(\alpha+\beta,-\eta;\alpha;\frac{k^m-z^m}{k^m-h^m}\right)$$

$$\cdot\prod_{i=1}^{n}\left\{{}_{p_i}F_{q_i}\left[\begin{matrix}(a^i_{p_i});\\(b^i_{q_i});\end{matrix}r_i(z^m-h^m)\right]\right\}\prod_{j=1}^{n'}\left\{{}_{A_j}F_{B_j}\left[\begin{matrix}(c^j_{A_j});\\(d^j_{B_j});\end{matrix}s_j(z^m-h^m)\right]\right\}d(z^m)$$

On a simple change of variables and then using (1.2.1),(3.7) is in the form

$$(3.8)\; E\{g(z)\}=\Im^{-1}\Gamma(\alpha)x^{m(\alpha+\beta)}\,I^{\alpha,\beta,\eta}_{0,x:m}\left[(x^m)^{p-1}\prod_{i=1}^{n}\left\{{}_{p_i}F_{q_i}\left[\begin{matrix}(a^i_{p_i});\\(b^i_{q_i});\end{matrix}r_ix^m\right]\right\}\prod_{j=1}^{n'}\left\{{}_{A_j}F_{B_j}\left[\begin{matrix}(c^j_{A_j});\\(d^j_{B_j});\end{matrix}s_jx^m\right]\right\}\right]$$

Where $k^m-h^m=x^m$

An application of Lemma 4 yields (3.5) where $\aleph$ is given by (3.6).

Theorem 3: If $f(z)$ defined by (3.1) represents a p.d.f. then its characteristic function $\phi(t)$ is given by

$$(3.9)\quad \phi(t)=\aleph^{-1}e^{ith^m}F^{2:p_1,\ldots,p_n}_{2:q_1,\ldots,q_n}\left[\begin{matrix}p,p-\beta+\eta & :(a^1_{p_1});\ldots;(a^n_{p_n});\\ p-\beta,p+\alpha+\eta & :(b^1_{q_1});\ldots;(b^n_{q_n});\end{matrix}r_1(k^m-h^m),\ldots,r_n(k^m-h^m),it(k^m-h^m)\right]$$

Provided $(k\neq h)$, $\operatorname{Re}(\alpha)>0,\beta,\eta\in\mathbb{C}$, $p-1\in A(\beta,\eta)$, $p_i\le q_i+1$

$\left(p_i=q_i+1,\max_{1\le i\le n}\left\{\left|r_i(k^m-h^m),it(k^m-h^m)\right|\right\}<1\right)\ \forall i\in\{1,\ldots,n\}$, $m\in\mathbb{N}$ and $\aleph$ is defined by (3.6).

Proof: For the p.d.f. $f(z)$ defined by (3.1) we have

$$\phi(t)=\int_{-\infty}^{\infty}e^{itz^m}f(z)d(z^m)$$

$$=\Im^{-1}\int_{h^m}^{k^m}e^{itz^m}(z^m-h^m)^{p-1}(k^m-z^m)^{\alpha-1}F\left(\alpha+\beta,-\eta;\alpha;\frac{k^m-z^m}{k^m-h^m}\right).$$

$$\prod_{i=1}^{n}\left\{{}_{p_i}F_{q_i}\left[\begin{matrix}(a^i_{p_i});\\(b^i_{q_i});\end{matrix}r_i(z^m-h^m)\right]\right\}d(z^m)$$

$$=\Im^{-1}\Gamma(\alpha)x^{m(\alpha+\beta)}e^{ith^m}I^{\alpha,\beta,\eta}_{0,x:m}\left[(x^m)^{p-1}e^{itx^m}\prod_{i=1}^{n}\left\{{}_{p_i}F_{q_i}\left[\begin{matrix}(a^i_{p_i});\\(b^i_{q_i});\end{matrix}r_ix^m\right]\right\}\right]$$

Where $k^m-h^m=x^m\quad(k\neq h)$

Now applying Lemma 3, we get the desired characteristic function as (3.9).

Remark: One of the necessary condition for $\phi(t)$ to be a c.f. is $\phi(0)=1$ which is satisfied by (3.9) and rest all the conditions are also satisfied by (3.9) which confirms that $\phi(t)$ given by (3.9) is c.f. of p.d.f. given by (3.1).

Theorem 4: If $f(z)$ defined by (3.1) represents a p.d.f. then the m.g.f. about origin is given by

(3.10) $$\underset{z=0}{M}(t)=\aleph^{-1}e^{th^m}F^{2,p_1,\ldots,p_n}_{2,q_1,\ldots,q_n}\left[\begin{matrix}\lambda,\lambda-\beta+\eta & :(a^1_{p_1});\ldots;(a^n_{p_n}); \\ \lambda-\beta,\lambda+\alpha+\eta & :(b^1_{q_1});\ldots;(b^n_{q_n});\end{matrix} r_1(k^m-h^m),\ldots,r_n(k^m-h^m),t(k^m-h^m)\right]$$

Under the same conditions as in (3.9) and $\aleph$ is given by (3.6).

Proof: For the p.d.f. $f(z)$ defined by (3.1), the m.g.f. about origin is

$$\underset{z=0}{M}(t)=\int e^{tz^m}f(z)d(z^m)$$

$$=\Im^{-1}\int_{h^m}^{k^m}e^{tz^m}(z^m-h^m)^{p-1}(k^m-z^m)^{\alpha-1}F\left(\alpha+\beta,-\eta;\alpha;\frac{k^m-z^m}{k^m-h^m}\right)\cdot\prod_{i=1}^{n}\left\{{}_{p_i}F_{q_i}\left[\begin{matrix}(a^i_{p_i}); \\ (b^i_{q_i});\end{matrix} r_i(z^m-h^m)\right]\right\}d(z^m)$$

For $h^m<z^m<k^m, h>0$

$$=\Im^{-1}\Gamma(\alpha)x^{m(\alpha+\beta)}e^{th^m}I^{\alpha,\beta,\eta}_{0,x;m}\left[(x^m)^{p-1}e^{tx^m}\prod_{i=1}^{n}\left\{{}_{p_i}F_{q_i}\left[\begin{matrix}(a^i_{p_i}); \\ (b^i_{q_i});\end{matrix} r_ix^m\right]\right\}\right]$$

Where $k^m-h^m=x^m \quad (k\neq h)$

Applying Lemma 3 we get the desired m.g.f. as (3.10).

Theorem 5: If $f(z)$ defined by (3.1) represents a p.d.f. and $h^m<t^m<k^m, h>0$ then the cummulative function $F(t)$ is given by

(3.11) $$F(t)=\aleph^{-1}\frac{\Gamma(p-\beta)\Gamma(\eta-\beta)\Gamma(p+\alpha+\eta)}{\Gamma(p+1)\Gamma(p+\eta-\beta)\Gamma(\alpha+\eta)\Gamma(-\beta)}\left(\frac{t^m-h^m}{k^m-h^m}\right)^p\cdot$$

$$F^{1:p_1,\ldots,p_n;2}_{1:q_1,\ldots,q_n;1}\left[\begin{matrix}p & :(a^1_{p_1});\ldots;(a^n_{p_n}); & -\eta-\alpha+1;\beta+1; \\ p+1 & :(b^1_{q_1});\ldots;(b^n_{q_n}); & \beta-\eta+1;\end{matrix} r_1(t^m-h^m),\ldots,r_n(t^m-h^m),\frac{t^m-h^m}{k^m-h^m}\right]$$

$$+\aleph^{-1}\frac{\Gamma(p-\beta)\Gamma(\beta-\eta)\Gamma(p+\alpha+\eta)}{\Gamma(p)\Gamma(p+\eta-\beta+1)\Gamma(\alpha+\beta)\Gamma(-\eta)}\left(\frac{t^m-h^m}{k^m-h^m}\right)^{p-\beta+\eta}\cdot$$

$$F^{1:p_1,\ldots,p_n;2}_{1:q_1,\ldots,q_n;1}\left[\begin{matrix}p-\beta+\eta & :(a^1_{p_1});\ldots;(a^n_{p_n}); & -\beta-\alpha+1,\eta+1; \\ p-\beta+\eta+1 & :(b^1_{q_1});\ldots;(b^n_{q_n}); & \eta-\beta+1;\end{matrix} r_1(t^m-h^m),\ldots,r_n(t^m-h^m),\frac{t^m-h^m}{k^m-h^m}\right]$$

Provided Re $(\alpha)>0, \beta,\eta\in\mathbb{C}$, $p-1\in A(\beta,\eta)$, $p_i\leq q_i$ and for

$$\left(p_i=q_i,\max_{1\leq i\leq n}\left\{r_i(t^m-h^m),\frac{t^m-h^m}{k^m-h^m}\right\}<1\right)\forall i\in\{1,\ldots,n\}, m\in\mathbb{N}$$

and $\aleph$ is given by (3.6).

Proof: For the p.d.f. $f(z)$ defined by (3.1) the cummulative probability function for the distribution function $F(t)$ is

$$F(t)=\int_{-\infty}^{t} f(z)d(z^m)$$

$$=\Im^{-1}\int_{h^m}^{t^m}(z^m-h^m)^{p-1}(k^m-z^m)^{\alpha-1}F\left(\alpha+\beta,-\eta;\alpha;\frac{k^m-z^m}{k^m-h^m}\right)$$

$$.\prod_{i=1}^{n}\left\{{}_{p_i}F_{q_i}\left[\begin{matrix}(a_{p_i}^i);\\(b_{q_i}^i);\end{matrix}r_i(z^m-h^m)\right]\right\}d(z^m)$$

(3.12) $$F(t)=\Im^{-1}\int_{0}^{t^m-h^m}(u^m)^{p-1}(x^m-u^m)^{\alpha-1}F\left(\alpha+\beta,-\eta;\alpha;\frac{x^m-u^m}{x^m}\right)$$

$$.\prod_{i=1}^{n}\left\{{}_{p_i}F_{q_i}\left[\begin{matrix}(a_{p_i}^i);\\(b_{q_i}^i);\end{matrix}r_iu^m\right]\right\}d(u^m)$$

Where $k^m-h^m=x^m$ and $z^m-h^m=u^m$

Making use of the contiguous function relation of the Gauss function [2]

$$F(a,b;c;z)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a,b;a+b-c+1;1-z)$$

$$+\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b}F(c-a,c-b;c-a-b+1;1-z),\ \left(|\arg(1-z)|<\pi\right)$$

And the Eulerian transformation

$$F(a,b;c;z)=(1-z)^{c-a-b}F(c-a,c-b;c;z),\quad\left(|\arg(1-z)|<\pi\right)$$

In the resulting R.H.S. of (3.12), expanding the Gauss function and then integrating term by term we get the desired cummulative function as (3.11).

4. Multivariate distribution

We consider the multivariate density function

(4.1) $$f(z_1^m,\ldots,z_n^m)=\omega(z_1^m)^{p_1-1}\ldots(z_n^m)^{p_n-1}(\Sigma z_i^m-\mu^m)^{p-1}(\rho^m-\Sigma z_i^m)^{\alpha-1}$$

$$.F\left(\alpha+\beta,-\eta;\alpha;\frac{\rho^m-\Sigma z_i^m}{\rho^m-\mu^m}\right).\prod_{i=1}^{n}\left\{{}_{p_i}F_{q_i}\left[\begin{matrix}(a_{p_i}^i);\\(b_{q_i}^i);\end{matrix}\Sigma z_i^m-\mu^m\right]\right\}$$

For $\mu^m<\Sigma z_i^m<\rho^m$

$=0$, otherwise

Where $\text{Re}\,(p_i) > 0, \forall i \in \{1,\ldots,n\}, m \in \mathbb{N}$, $\text{Re}(p) > 0$,

$\alpha, \beta, \eta \in \mathbb{C}, p-1 \in A(\beta,\eta)$ and

(4.2)
$$\omega^{-1} = B(p_1,\ldots,p_n)\frac{\Gamma(\alpha)\Gamma(p)\Gamma(p-\beta+\eta)}{\Gamma(p-\beta)\Gamma(p+\alpha+\eta)}v^{m(p+\alpha-1)}(\mu^m)^{\Sigma p_i-1}$$
$$.F^{2:p_1,\ldots,p_n;1}_{2:q_1,\ldots,q_n;\_}\left[\begin{matrix} p, p-\beta+\eta & : & (a^1_{p_1});\ldots;(a^n_{p_n});1-\Sigma p_i; \\ p-\beta, p+\alpha+\eta & : & (b^1_{q_1});\ldots;(b^n_{q_n});\_; \end{matrix} v^m,\ldots,v^m,\frac{v^m}{\mu^m}\right]$$

Where $\rho^m - \mu^m = v^m$

Theorem 6: If (4.1) is a p.d.f. then

$$\int_{\Omega}\cdots\int f(z_1^m,\ldots,z_n^m)d(z_1^m),\ldots,d(z_n^m) = 1$$

Proof:
$$\int_{\Omega}\cdots\int f(z_1^m,\ldots,z_n^m)d(z_1^m),\ldots,d(z_n^m) = \omega\int\cdots\int_{z_i^m>0,\mu^m\le\Sigma z_i^m\le\rho^m}(z_1^m)^{p_1-1}\ldots(z_n^m)^{p_n-1}$$
$$(\Sigma z_i^m-\mu^m)^{p-1}(\rho^m-\Sigma z_i^m)^{\alpha-1}.F\left(\alpha+\beta,-\eta;\alpha;\frac{\rho^m-\Sigma z_i^m}{\rho^m-\mu^m}\right).\prod_{i=1}^n\left\{{}_{p_i}F_{q_i}\left[\begin{matrix}(a^i_{p_i});\\(b^i_{q_i});\end{matrix}\Sigma z_i^m-\mu^m\right]\right\}$$
$$d(z_1^m),\ldots,d(z_n^m)$$
$$=\omega B(p_1,\ldots,p_n)\int_{\mu^m}^{\rho^m}(z^m)^{\Sigma p_i-1}(z^m-\mu^m)^{p-1}(\rho^m-z^m)^{\alpha-1}.F\left(\alpha+\beta,-\eta;\alpha;\frac{\rho^m-z^m}{\rho^m-\mu^m}\right)$$
$$.\prod_{i=1}^n\left\{{}_{p_i}F_{q_i}\left[\begin{matrix}(a^i_{p_i});\\(b^i_{q_i});\end{matrix}z^m-\mu^m\right]\right\}d(z^m)$$
$$=\omega B(p_1,\ldots,p_n)\;\Gamma(\alpha)v^{m(\alpha+\beta)}I^{\alpha,\beta,\eta}_{0,v,m}\left[(v^m)^{p-1}(v^m+\mu^m)^{\Sigma p_i-1}\prod_{i=1}^n\left\{{}_{p_i}F_{q_i}\left[\begin{matrix}(a^i_{p_i});\\(b^i_{q_i});\end{matrix}v^m\right]\right\}\right]$$

Where $\rho^m - \mu^m = v^m$

Applying Lemma 2 we get

$\int_{\Omega}\cdots\int f(z_1^m,\ldots,z_n^m)d(z_1^m),\ldots,d(z_n^m) = 1$ which shows that (4.1) is a p.d.f.

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Parabolic Starlike and Uniformly Convex Functions

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Abstract

The main object of this paper is to derive the sufficient conditions for the function $z\left\{{}_p\psi_q\left(z\right)\right\}$ to be in the class of uniformly starlike and uniformly convex function associated with the parabolic region $\text{Re}\left\{\omega\right\} > |\omega - 1|$. Further, the hadamard product of the function which are analytic in the open unit disk with negative coefficients are also investigated. Finally, similar results using an integral operator are also obtained.

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Keywords: Analytic Functions, Univalent Functions, Convex Functions, Starlike Functions, Fox Wright Function, Integral Operator, Parabolic Region

1. Introduction

Let T denote the class of function $f(z)$ of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad , \quad a_n \geq 0 \tag{1.1}$$

that are analytic in the open unit disk $U = \{z : |z| < 1\}$

A function $f \in T$ is said to be starlie of order $\alpha, 0 \leq \alpha < 1$, if and only if $\text{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha$ and convex if and only if $\text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha$, with $z \in U$

Definition: 1.1 A function $f(z)$ is uniformly convex (uniformly starlike) in U if $f(z)$ is convex (starlike) and has the property that for every circular arc ξ contained in U, with center ξ also in U ,the arc $f(\xi)$is convex (starlike) with respect to $f(\xi)$.The class of uniformly convex functions is denoted by UC_p and the class of uniformly starlike functions by US_p [5] (also refer [7],[10]).
These subclasses were introduced and defined by Goodman [2] and [3]. The class UC_p describes geometrically the domain of values of the expression $\left(1+\frac{zf\prime\prime(z)}{f\prime(z)}\right)$,$z \in Uf(z) \in UC_p$ as a parabolic region [8]

$$\Re = \left\{\omega \in C : (\mathrm{Im}\omega)^2 < 2\mathrm{Re}\omega - 1\right\}$$

Definition: 1.2 [12] $f \in UC_p(\lambda, \mu)$ if and only if

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) \geq \left|\frac{zf''(z)}{f'(z)}\right|, z \in U \tag{1.2}$$

Definition: 1.3 Let $f \in T, 0 \leq \lambda < \infty$ and $0 \leq \mu < 1$, then $f \in UC_p(\lambda, \mu)$ if and only if

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) \geq \lambda\left|\frac{zf''(z)}{f'(z)}\right| + \mu \tag{1.3}$$

The following sufficient condition on the coefficient of the class $UC_p(\lambda, \mu)$ was obtained in [9].
Definition: 1.4 [4] Let $0 \leq \lambda < \infty$ and $0 \leq \mu < 1$, then $f \in US_p(\lambda, \mu)$ if and only if

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) \geq \lambda\left|\frac{zf'(z)}{f(z)} - 1\right| + \mu \tag{1.4}$$

Definition: 1.5 For function $f(z)$ of the form(1.1) and $g(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n$ we define the hadamard product (or convolution) of f and g by

$$(f * g)(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n, z \in U \tag{1.5}$$

Definition: 1.6 [6] The Fox-Wright function appearing in the present paper is defined by

$${}_p\psi_q(z) = {}_p\psi_q\left[\begin{array}{cc} (a_j, \alpha_j)_{1,p}; & z \\ (b_j, \beta_j)_{1,q}; & \end{array}\right] = \sum_{n=0}^{\infty} \frac{\prod\limits_{j=1}^{p} \Gamma(a_j + \alpha_j n) z^n}{\prod\limits_{j=1}^{q} \Gamma(b_j + \beta_j n) n!} \tag{1.6}$$

where $\alpha_j\,(j = 1, \ldots, p)$ and $\beta_j\,(j = 1, \ldots, q)$ are real and positive and

$$1 + \sum_{j=1}^{q} \beta_j > \sum_{j=1}^{p} \alpha_j$$

Definition: 1.7 For ${}_p\Psi_q$ given by (1.6) we have

$$z_p\psi_q(z) = \sum_{n=1}^{\infty} \frac{\prod_{j=1}^{p} \Gamma\left(a_j + \alpha_j\left(n-1\right)\right) z^n}{\prod_{j=1}^{q} \Gamma\left(b_j + \beta_j\left(n-1\right)\right)\left(n-1\right)!}$$

$$= \frac{\prod_{j=1}^{p} \Gamma a_j}{\prod_{j=1}^{q} \Gamma b_j} z + \sum_{n=2}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(a_j + a_j(n-1))\, z^n}{\prod_{j=1}^{q} \Gamma(b_j + \beta_j(n-1))\,(n-1)!}$$

$$z\frac{\prod_{j=1}^{q} \Gamma b_j}{\prod_{j=1}^{p} \Gamma a_j}{}_p\psi_q = z_p\Omega_q = z + \sum_{n=2}^{\infty} \frac{\prod_{j=1}^{q} \Gamma b_j}{\prod_{j=1}^{p} \Gamma a_j} \frac{\prod_{j=1}^{p} \Gamma\left(a_j + \alpha_j\left(n-1\right)\right) z^n}{\prod_{j=1}^{q} \Gamma\left(b_j + \beta_j\left(n-1\right)\right)\left(n-1\right)!} \tag{1.7}$$

where
$${}_p\Omega_q = \frac{\prod_{j=1}^{q} \Gamma b_j}{\prod_{j=1}^{p} \Gamma a_j}{}_p\psi_q$$

Definition: 1.8 Let $G_q^p : T \to T$ be a linear operator defined by

$$G_q^p f\left(z\right) = z_p\Omega_q * f\left(z\right)$$

$$= z - \sum_{n=2}^{\infty} \frac{\prod_{j=1}^{q} \Gamma b_j}{\prod_{j=1}^{p} \Gamma a_j} \frac{\prod_{j=1}^{p} \Gamma\left(a_j + \alpha_j\left(n-1\right)\right)}{\prod_{j=1}^{q} \Gamma\left(b_j + \beta_j\left(n-1\right)\right)} a_n \frac{z^n}{\left(n-1\right)!}$$

$$G_q^p f\left(z\right) = z - \sum_{n=2}^{\infty} \mathrm{B}\left(a_j, n\right) a_n z^n \tag{1.8}$$

where

$$\mathrm{B}\left(a_j, n\right) = \frac{\prod_{j=1}^{q} \Gamma b_j}{\prod_{j=1}^{p} \Gamma a_j} \frac{\prod_{j=1}^{p} \Gamma\left(a_j + \alpha_j\left(n-1\right)\right)}{\prod_{j=1}^{q} \Gamma\left(b_j + \beta_j\left(n-1\right)\right)} \tag{1.9}$$

Definition: 1.9 For $-1 \le \gamma < 1$, we let $S_q^p(\gamma)$ denote the subclass of starlike functions corresponding to the family UC_p for $f(z)$ s.t.

$$\operatorname{Re}\left\{ \frac{z\left(G_q^p f\left(z\right)\right)'}{G_q^p f\left(z\right)} - \gamma \right\} \ge \left| \frac{z\left(G_q^p f\left(z\right)\right)'}{G_q^p f\left(z\right)} - 1 \right| \tag{1.10}$$

Lemma (1.1) A function $f(z)$ of the form (1.1) is in $US_p(\lambda,\mu)$ if and only if

$$\sum_{n=2}^{\infty}[n\,(1+\lambda)-(\lambda+\mu)]a_n \le (1-\mu)\,M_1 \qquad (1.11)$$

where $M_1>0$ is a suitable constant

Lemma (1.2) [1] A function $f(z)$ of the form (1.1) is in $f\in UC_p(\lambda,\mu)$ if it satisfies the condition

$$\sum_{n=2}^{\infty}n\,[n\,(1+\lambda)-(\lambda+\mu)]a_n \le (1-\mu)\,M_2 \qquad (1.12)$$

where $M_2>0$is a suitable constant.

2. Main Results

Theorem2.1 If $\sum\limits_{j=1}^{q} b_j > \sum\limits_{j=1}^{p} a_j+1, a_j>0$ and $1+\sum\limits_{j=1}^{q}\beta_j > \sum\limits_{j=1}^{p}\alpha_j$ then a sufficient condition for the function $z\left\{{}_p\psi_q\left(z\right)\right\}$ to be in the class $US_p\left(\lambda,\mu\right)$, $0\le\lambda<\infty$ and $0\le\lambda<\infty$ is

$$\frac{(1+\lambda)}{(1-\mu)}{}_p\psi_q\left[\begin{array}{cc}(a_j+\alpha_j,\alpha_j)_{1,p}; & 1\\ (b_j+\beta_j,\beta_j)_{1,q}; & \end{array}\right]+{}_p\psi_q\left[\begin{array}{cc}(a_j,\alpha_j)_{1,p}; & 1\\ (b_j,\beta_j)_{1,q}; & \end{array}\right]\le M_1+\frac{\prod\limits_{j=1}^{p}\Gamma a_j}{\prod\limits_{j=1}^{q}\Gamma b_j} \qquad (2.1)$$

Proof: Since

$$z_p\psi_q(z)=z\sum_{n=0}^{\infty}\frac{\prod\limits_{j=1}^{p}\Gamma\left(a_j+\alpha_j n\right)z^n}{\prod\limits_{j=1}^{q}\Gamma\left(b_j+\beta_j n\right)n!}$$

$$=\frac{\prod\limits_{j=1}^{p}\Gamma a_j}{\prod\limits_{j=1}^{q}\Gamma b_j}z+\sum_{n=2}^{\infty}\frac{\prod\limits_{j=1}^{p}\Gamma\left(a_j+\alpha_j(n-1)\right)z^n}{\prod\limits_{j=1}^{q}\Gamma\left(b_j+\beta_j(n-1)\right)(n-1)!}$$

So by the virtue of Lemma 1.1, we need only to show that

$$\sum_{n=2}^{\infty}[n\,(1+\lambda)-(\lambda+\mu)]\left[\frac{\prod\limits_{j=1}^{p}\Gamma\left(a_j+\alpha_j(n-1)\right)}{\prod\limits_{j=1}^{q}\Gamma\left(b_j+\beta_j(n-1)\right)(n-1)!}\right]\le(1-\mu)\,M_1$$

Now,

$$\sum_{n=2}^{\infty}[n\,(1+\lambda)-(\lambda+\mu)]\left[\frac{\prod\limits_{j=1}^{p}\Gamma\left(a_j+\alpha_j(n-1)\right)}{\prod\limits_{j=1}^{q}\Gamma\left(b_j+\beta_j(n-1)\right)(n-1)!}\right]$$

$$= \sum_{n=0}^{\infty} [(n+2)(1+\lambda) - (\lambda+\mu)] \left[\frac{\prod_{j=1}^{p} \Gamma(a_j + \alpha_j(n+1))}{\prod_{j=1}^{q} \Gamma(b_j + \beta_j(n+1))\,(n+1)!} \right]$$

$$= \sum_{n=0}^{\infty} (n+1)(1+\lambda) \left[\frac{\prod_{j=1}^{p} \Gamma(a_j + \alpha_j(n+1))}{\prod_{j=1}^{q} \Gamma(b_j + \beta_j(n+1))\,(n+1)!} \right] + \sum_{n=0}^{\infty} (1-\mu) \left[\frac{\prod_{j=1}^{p} \Gamma(a_j + \alpha_j(n+1))}{\prod_{j=1}^{q} \Gamma(b_j + \beta_j(n+1))\,(n+1)!} \right]$$

$$= \sum_{n=0}^{\infty} (1+\lambda) \left[\frac{\prod_{j=1}^{p} \Gamma(a_j + \alpha_j(n+1))}{\prod_{j=1}^{q} \Gamma(b_j + \beta_j(n+1))\,n!} \right] + (1-\mu) \left[\sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(a_j + \alpha_j n)}{\prod_{j=1}^{q} \Gamma(b_j + \beta_j n)\,n!} - \frac{\prod_{j=1}^{p} \Gamma a_j}{\prod_{j=1}^{q} \Gamma b_j} \right]$$

$$(1+\lambda)\,{}_p\psi_q \left[\begin{array}{cc} (a_j+\alpha_j, \alpha_j)_{1,p}; & \\ (b_j+\beta_j, \beta_j)_{1,q}; & 1 \end{array} \right] + (1-\mu)\,{}_p\psi_q \left[\begin{array}{cc} (a_j, \alpha_j)_{1,p}; & \\ (b_j, \beta_j)_{1,q}; & 1 \end{array} \right] - (1-\mu) \frac{\prod_{j=1}^{p} \Gamma a_j}{\prod_{j=1}^{q} \Gamma b_j}$$

$$= (1-\mu) \left\{ \frac{(1+\lambda)}{(1-\mu)} {}_p\psi_q \left[\begin{array}{cc} (a_j+\alpha_j, \alpha_j)_{1,p}; & \\ (b_j+\beta_j, \beta_j)_{1,q}; & 1 \end{array} \right] + {}_p\psi_q \left[\begin{array}{cc} (a_j, \alpha_j)_{1,p}; & \\ (b_j, \beta_j)_{1,q}; & 1 \end{array} \right] - \frac{\prod_{j=1}^{p} \Gamma a_j}{\prod_{j=1}^{q} \Gamma b_j} \right\}$$

This expression is bounded above by $(1-\mu)\,M_1$, if and only if

$$= \frac{(1+\lambda)}{(1-\mu)} {}_p\psi_q \left[\begin{array}{cc} (a_j+\alpha_j, \alpha_j)_{1,p}; & \\ (b_j+\beta_j, \beta_j)_{1,q}; & 1 \end{array} \right] + {}_p\psi_q \left[\begin{array}{cc} (a_j, \alpha_j)_{1,p}; & \\ (b_j, \beta_j)_{1,q}; & 1 \end{array} \right] \le M_1 + \frac{\prod_{j=1}^{p} \Gamma a_j}{\prod_{j=1}^{q} \Gamma b_j}$$

Remark 2.1 In Theorem 2.1 $\lambda = 2$ and $\mu = 0$ gives the sufficient condition for the function $z\{{}_p\psi_q(z)\}$ to be in the class $US_p(2,0), 0 \le \lambda < \infty$ and $0 \le \mu < 1$ obtained in [11].
We get this similar result for $\alpha = 0$ in [11, Th 2.2] i.e.

$$= 3\,{}_p\psi_q \left[\begin{array}{cc} (a_j+\alpha_j, \alpha_j)_{1,p}; & \\ (b_j+\beta_j, \beta_j)_{1,q}; & 1 \end{array} \right] + {}_p\psi_q \left[\begin{array}{cc} (a_j, \alpha_j)_{1,p}; & \\ (b_j, \beta_j)_{1,q}; & 1 \end{array} \right] \le M_1 + \frac{\prod_{j=1}^{p} \Gamma a_j}{\prod_{j=1}^{q} \Gamma b_j}$$

Theorem 2.2 Let $f(z)$ be given by (1.1). if $-1 \le \gamma < 1$ and

$$(2.2) \sum_{n=2}^{\infty} (2n-1-\gamma)\, B(a_j, n)\, |a_n| \le 1-\gamma$$

then $f(z) \in S_q^p(\gamma)$ where $B(a_j, n)$ is given by (1.9).
Proof: By Def (1.9) of the class $S_q^p(\gamma)$, it suffices to show that

$$\operatorname{Re}\left\{\frac{z\left(G_q^p f(z)\right)'}{G_q^p f(z)} - \gamma\right\} \geq \left|\frac{z\left(G_q^p f(z)\right)'}{G_q^p f(z)} - 1\right| \tag{2.3}$$

$$\text{or} \qquad 2\left|\frac{z\left(G_q^p f(z)\right)'}{G_q^p f(z)} - 1\right| \leq 1 - \gamma$$

$$\text{or} \quad 2\left|\frac{z\left\{1 - \sum\limits_{n=2}^{\infty} B(a_j, n)\, a_n n z^{n-1}\right\}}{z - \sum\limits_{n=2}^{\infty} B(a_j, n)\, a_n z^n} - 1\right| \leq 1 - \gamma$$

$$\text{or} \quad 2\sum_{n=2}^{\infty} B(a_j, n)(n-1)|a_n| \leq (1-\gamma)\left\{1 - \sum_{n=2}^{\infty} B(a_j, n)|a_n|\right\}$$

$$\text{or} \quad \sum_{n=2}^{\infty} (2n - 1 - \gamma) B(a_j, n)|a_n| \leq 1 - \gamma$$

Hence, the theorem is proved.

3. An Integral Operator

In this section we illustrate some results obtained by a particular integral operator defined by

$${}_p\phi_q\left[\begin{array}{ll} (a_j, \alpha_j)_{1,p}; & z \\ (b_j, \beta_j)_{1,q}; & \end{array}\right] = \int_0^z {}_p\psi_q(x)dx$$

Theorem 3.1 If $\sum\limits_{j=1}^{q} b_j > \sum\limits_{j=1}^{p} a_j, a_j > 0$ and $1 + \sum\limits_{j=1}^{q} \beta_j > \sum\limits_{j=1}^{p} \alpha_j$ then a sufficient condition for the function ${}_p\phi_q(z) = \int\limits_0^z {}_p\psi_q(x)dx$ to be in the class $US_p(\lambda, \mu)$ $0 \leq \lambda < \infty$ and $0 \leq \mu < 1$ is

$$\frac{(1+\lambda)}{(1-\mu)} {}_p\psi_q\left[\begin{array}{ll} (a_j, \alpha_j)_{1,p}; & 1 \\ (b_j, \beta_j)_{1,q}; & \end{array}\right] - \frac{(\lambda+\mu)}{(1-\mu)} {}_p\psi_q\left[\begin{array}{ll} (a_j - \alpha_j, \alpha_j)_{1,p}; & 1 \\ (b_j - \beta_j, \beta_j)_{1,q}; & \end{array}\right]$$
$$+ \frac{(\lambda+\mu)}{(1-\mu)} \frac{\prod\limits_{j=1}^{p} \Gamma(a_j - \alpha_j)}{\prod\limits_{j=1}^{q} \Gamma(b_j - \beta_j)} \leq M_1 + \frac{\prod\limits_{j=1}^{p} \Gamma a_j}{\prod\limits_{j=1}^{q} \Gamma b_j} \tag{3.1}$$

Proof: Since ${}_p\phi_q(z) = \int_0^z {}_p\psi_q(x)dx$

$$= \frac{\prod_{j=1}^{p} \Gamma a_j}{\prod_{j=1}^{q} \Gamma b_j} z + \sum_{n=2}^{\infty} \frac{\prod_{j=1}^{p} \Gamma\left[(a_j-\alpha_j)+\alpha_j n\right] z^n}{\prod_{j=1}^{q} \Gamma\left[(b_j-\beta_j)+\beta_j n\right] n!} \tag{3.2}$$

We see that

$$\sum_{n=2}^{\infty}\left[n(1+\lambda)-(\lambda+\mu)\right]\frac{\prod_{j=1}^{p} \Gamma\left[(a_j-\alpha_j)+\alpha_j n\right]}{\prod_{j=1}^{q} \Gamma\left[(b_j-\beta_j)+\beta_j n\right] n!}$$

$$= (1+\lambda)\sum_{n=2}^{\infty}\frac{n\prod_{j=1}^{p} \Gamma\left[(a_j-\alpha_j)+\alpha_j n\right]}{\prod_{j=1}^{q} \Gamma\left[(b_j-\beta_j)+\beta_j n\right] n!} - (\lambda+\mu)\sum_{n=2}^{\infty}\frac{\prod_{j=1}^{p} \Gamma\left[(a_j-\alpha_j)+\alpha_j n\right]}{\prod_{j=1}^{q} \Gamma\left[(b_j-\beta_j)+\beta_j n\right] n!}$$

$$= (1+\lambda)\left[\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(a_j+\alpha_j n)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j n)n!} - \frac{\prod_{j=1}^{p}\Gamma a_j}{\prod_{j=1}^{q}\Gamma b_j}\right](\lambda+\mu)\left[\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma[(a_j-\alpha_j)+\alpha_j n]}{\prod_{j=1}^{q}\Gamma[(b_j-\beta_j)+\beta_j n]n!} - \frac{\prod_{j=1}^{p}\Gamma(a_j-\alpha_j)}{\prod_{j=1}^{q}\Gamma(b_j-\beta_j)} - \frac{\prod_{j=1}^{p}\Gamma a_j}{\prod_{j=1}^{q}\Gamma b_j}\right]$$

$$= (1+\lambda)\,{}_p\psi_q\left[\begin{array}{cc}(a_j,\alpha_j)_{1,p}; & 1\\ (b_j,\beta_j)_{1,q}; & \end{array}\right] - (\lambda+\mu)\,{}_p\psi_q\left[\begin{array}{cc}(a_j-\alpha_j,\alpha_j)_{1,p}; & 1\\ (b_j-\beta_j,\beta_j)_{1,q}; & \end{array}\right]$$

$$+(\lambda+\mu)\frac{\prod_{j=1}^{p}\Gamma(a_j-\alpha_j)}{\prod_{j=1}^{q}\Gamma(b_j-\beta_j)} - (1-\mu)\frac{\prod_{j=1}^{p}\Gamma a_j}{\prod_{j=1}^{q}\Gamma b_j}$$

$$= (1-\mu)\left\{\frac{(1+\lambda)}{(1-\mu)}{}_p\psi_q\left[\begin{array}{cc}(a_j,\alpha_j)_{1,p}; & 1\\ (b_j,\beta_j)_{1,q}; & \end{array}\right] - \frac{(\lambda+\mu)}{(1-\mu)}{}_p\psi_q\left[\begin{array}{cc}(a_j-\alpha_j,\alpha_j)_{1,p}; & 1\\ (b_j-\beta_j,\beta_j)_{1,q}; & \end{array}\right]\right.$$

$$\left. + \frac{(\lambda+\mu)}{(1-\mu)}\frac{\prod_{j=1}^{p}\Gamma(a_j-\alpha_j)}{\prod_{j=1}^{q}\Gamma(b_j-\beta_j)} - \frac{\prod_{j=1}^{p}\Gamma a_j}{\prod_{j=1}^{q}\Gamma b_j}\right\}$$

$$\leq (1-\mu)M_1$$

where M_1 is a constant greater than the expression in (3.2) which in view of Lemma 1.1 gives the desired result (3.1).
Remark: For $\alpha = 2$ and $\mu = 0$ theorem (3.1) reduce to the desired result obtained by Chaurasia and Srivastava.

Theorem (3.2) If $\sum_{j=1}^{q} b_j > \sum_{j=1}^{p} a_j, a_j > 0$ and $1+\sum_{j=1}^{q} \beta_j > \sum_{j=1}^{p} \alpha_j$ then a sufficient condition for the function ${}_p\phi_q(z) = \int_0^z {}_p\psi_q(x)dx$ to be in the class $UC_p(\lambda,\mu), 0 \le \lambda < \infty$ and $0 \le \mu < 1$ is

$$\frac{(1+\lambda)}{(1-\mu)} {}_p\psi_q\left[\begin{array}{ll} (a_j+\alpha_j,\alpha_j)_{1,p}; & 1 \\ (b_j+\beta_j,\beta_j)_{1,q}; & \end{array}\right] + {}_p\psi_q\left[\begin{array}{ll} (a_j,\alpha_j)_{1,p}; & 1 \\ (b_j,\beta_j)_{1,q}; & \end{array}\right] \le M_2 + \frac{\prod_{j=1}^{p}\Gamma a_j}{\prod_{j=1}^{q}\Gamma b_j} \tag{3.3}$$

Proof: The proof of this theorem is a direct consequence of Th.3.1

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COEFFICIENT INEQUALITIES OF MEROMORPHIC MULTIVALENT STARLIKE AND CONVEX FUNCTIONS

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ABSTRACT: This paper envisages some coefficient properties of functions $f(z)$ belong to the class $\Sigma_r(p)$ of meromorphic multivalent functions in the punctured disk D_r. Motivated by several earlier works, we consider two subclasses $S_r^*(\alpha)$ and $C_r(\alpha)$ of $\Sigma_r(p)$.Further we discuss the starlikeness and the convexity of functions $f(z)$ in $\Sigma_r(p)$ .These results are of more importance as they generalize the results of Shigeyoshi Owa, Nicole N. Pascu, M.L.Mogra, T.R.Reddy and O.P.Juneja et al.

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1. Introduction

Let $\Sigma_r(p)$ denote the class of functions $f(z)$ of the form:

$$f(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} a_{n+p-1} z^{n+p-1}$$

which are analytic in the punctured disk $D_r = \{ z \in C : 0 < |z| = r \le 1 \}$

A function $f(z) \in \Sigma_r(p)$ is said to be starlike of order α if it satisfies the inequality:

$$\text{Re}\left(-\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in D_r) \tag{1.2}$$

for some $\alpha\left(0 \le \alpha < 1\right)$. We say that $f(z)$ is in the class $S_r^*(\alpha)$ for such functions.

A function $f(z) \in \Sigma_r(p)$ is said to be convex of order α if it satisfies the inequality:

$$\text{Re}\left\{-\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} > \alpha \qquad (z \in D_r) \tag{1.3}$$

for some $\alpha\left(0 \le \alpha < 1\right)$. We say that $f(z)$ is in the class $C_r(\alpha)$ if it is convex of order α in D_r.

We note that $f(z) \in C_r(\alpha)$ if and only if $-zf'(z) \in S_r^*(\alpha)$.

Ozaki [5] has shown that the necessary and sufficient condition that $f(z) \in \Sigma_r$ with $a_n \ge 0$ $(n = 1, 2, 3, \ldots)$ is meromorphic and univalent in D_r is that there should exist the relation:

$$\sum_{n=1}^{\infty} n a_n r^{n+1} \le 1$$

between its coefficients.

Owa and Pascu [4] gave an improved and extended result for the above theorem by Ozaki.

Now in the present paper we give some results on multivalent functions.

2. Coefficients Inequalities for functions

Theorem 2.1. *If* $f(z) \in \Sigma_r(p)$ *satisfies*

$$\sum_{n=0}^{\infty}\{n-1+p(1+\alpha)\}\left|a_{n+p-1}\right| r^{n+2p-1} \le p(1-\alpha) \tag{2.1}$$

for some $\alpha\left(0 \le \alpha < 1\right)$ *and* $k(\alpha < k \le 1)$ *then* $f(z) \in S_r^*(\alpha)$.

Proof: For $f(z) \in \Sigma_r(p)$ we know that

$$\left|zf'(z)+kpf(z)\right| - \left|zf'(z)+(2\alpha-k)pf(z)\right|$$

$$=\left|\frac{(k-1)p}{z^p}+\sum_{n=0}^{\infty}a_{n+p-1}[n+p(k+1)-1]z^{n+p-1}\right|$$

$$-\left|\frac{(2\alpha-k-1)p}{z^p}+\sum_{n=0}^{\infty}a_{n+p-1}[n+p-1+(2\alpha-k)p]z^{n+p-1}\right|$$

$$\le |(k-1)|\,pr^{-p}+\sum_{n=0}^{\infty}[n+p(k+1)-1]\left|a_{n+p-1}\right|r^{n+p-1}$$

$$+\sum_{n=0}^{\infty}[n+p-1+(2\alpha-k)p]\left|a_{n+p-1}\right|r^{n+p-1}-|k+1-2\alpha|pr^{-p}$$

$$=\left[1-k-(k+1-2\alpha)\right]pr^{-p}+\sum_{n=0}^{\infty}[n+p(k+1)-1+n+p-1+(2\alpha-k)p]\left|a_{n+p-1}\right|r^{n+p-1}$$

$$=r^{-p}\left[2(\alpha-k)p+\sum_{n=0}^{\infty}2\{n-1+p(1+\alpha)\}\left|a_{n+p-1}\right|r^{n+2p-1}\right]$$

$$\le 0$$

which shows that

$$\sum_{n=0}^{\infty}\{n-1+p(1+\alpha)\}\left|a_{n+p-1}\right|r^{n+2p-1}\le p(k-\alpha)\le p(1-\alpha)$$

It follows

$$|zf'(z)+kpf(z)|-|zf'(z)+(2\alpha-k)pf(z)|\le 0$$

So that

$$\mathrm{Re}\left(-\frac{zf'(z)}{f(z)}\right)>\alpha p\Rightarrow f(z)\in S_r^*(\alpha p)$$

Here we define a new subclass $S_r^*(\alpha p)$ for which $S_r^*(\alpha p)\subset S_r^*(\alpha)$

Therefore if $\quad f(z)\in S_r^*(\alpha p)$

Then $\quad f(z)\in S_r^*(\alpha) \qquad\qquad (z\in D_r)$

Theorem 2.2 *If* $f(z) \in \Sigma_r(p)$ *then* $f(z) \in S_r^*(\alpha p)$ *if and only if (2.1) is satisfied.*

Proof: In view of Theorem (2.1) it is sufficient to show the "only if" part.

Let $$f(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} a_{n+p-1} z^{n+p-1} \quad , a_{n+p-1} \geq 0 \text{ is in } S_r^*(\alpha p).$$

Then

$$\left| \frac{\frac{zf'(z)}{f(z)} + kp}{\frac{zf'(z)}{f(z)} + (2\alpha - k)p} \right| = \left| \frac{(k-1)p + \sum_{n=0}^{\infty} a_{n+p-1}[n + p(k+1) - 1] z^{n+2p-1}}{(2\alpha - k - 1)p + \sum_{n=0}^{\infty} a_{n+p-1}[n + p - 1 + (2\alpha - k)p] z^{n+2p-1}} \right| \leq 1$$

for all $(z \in D_r)$.

Taking real values and putting $|z| = r$ we obtain

$$(1-k)p + \sum_{n=0}^{\infty} [n + p(k+1) - 1] |a_{n+p-1}| r^{n+2p-1} \leq (k + 1 - 2\alpha)p - \sum_{n=0}^{\infty} [n + p - 1 + (2\alpha - k)p] |a_{n+p-1}| r^{n+2p-1}$$

or

$$\sum_{n=0}^{\infty} \{n - 1 + p(1+\alpha)\} |a_{n+p-1}| r^{n+2p-1} \leq p(k - \alpha) \leq p(1 - \alpha)$$

and hence the result follows.

Theorem 2.3. *If* $f(z) \in \Sigma_r(p)$ *satisfies*

(2.2) $$\sum_{n=0}^{\infty} n\{(n-1) + p(1+\alpha)\} |a_{n+p-1}| r^{n+2p-1} \leq p(1-\alpha)$$

for some $\alpha (0 \leq \alpha < 1)$ *then* $f(z) \in C_r(\alpha)$.

Proof: Noting that $f(z) \in C_r(\alpha)$ if and only if $-zf'(z) \in S_r^*(\alpha)$ and that

$$-zf'(z) = \frac{p}{z^p} - \sum_{n=0}^{\infty} (n + p - 1) a_{n+p-1} z^{n+p-1}$$

We complete the proof of the theorem with the aid of Theorem (2.1).

Theorem 2.4. *A function* $f(z) \in \Sigma_r(p)$ *belongs to* $S_r^*(\alpha)$ *for* $0 \le r \le r_0$ *where* r_0 *is the smallest positive root of the equation*

(2.3)

$$\left[p(1+\alpha)-1\right]\left|a_{p-1}\right|r^{2p+1}-\left[\delta\sqrt{\{r^2(1-p)+p\}}\,.r^{2p}+r^2p(1-\alpha)\right]-\left[p(1+\alpha)-1\right]\left|a_{p-1}\right|r^{2p-1}+p(1-\alpha)=0$$

and

$$\delta=\sqrt{\sum_{n=1}^{\infty}(n+p-1)\left|a_{n+p-1}\right|^2}+p\alpha\sqrt{\sum_{n=1}^{\infty}\frac{1}{(n+p-1)}\left|a_{n+p-1}\right|^2} \tag{2.4}$$

Proof: Using Cauchy Inequality, we see that

$$\sum_{n=0}^{\infty}\{(n-1)+p(1+\alpha)\}\left|a_{n+p-1}\right|r^{n+2p-1}$$

$$=\left[-1+p(1+\alpha)\right]\left|a_{p-1}\right|r^{2p-1}+\sum_{n=1}^{\infty}\{(n-1)+p(1+\alpha)\}\left|a_{n+p-1}\right|r^{n+2p-1}$$

$$\le\left[p(1+\alpha)-1\right]\left|a_{p-1}\right|r^{2p-1}+\sqrt{\sum_{n=1}^{\infty}(n+p-1)\left|a_{n+p-1}\right|^2}\sqrt{\sum_{n=1}^{\infty}(n+p-1)r^{2n+4p-2}}$$

$$+p\alpha\sqrt{\sum_{n=1}^{\infty}\frac{1}{(n+p-1)}\left|a_{n+p-1}\right|^2}\sqrt{\sum_{n=1}^{\infty}(n+p-1)r^{2n+4p-2}}$$

$$=\left[p(1+\alpha)-1\right]\left|a_{p-1}\right|r^{2p-1}+\frac{r^{2p}\sqrt{p+r^2(1-p)}}{(1-r^2)}\delta$$

$$\le p(1-\alpha)$$

where δ is given by (2.4). Therefore an application of Theorem (2.1) gives us that $f(z) \in S_r^*(\alpha)$ for $0 \le r \le r_0$.

Example 1: If we consider the function $f(z)$ given by

$$f(z)=\frac{1}{z^p}+\sum_{n=1}^{\infty}\frac{1}{(n+p-1)\sqrt{(n+p-1)}}e^{i\theta_{n+p-1}}z^{n+p-1} \qquad (\theta_{n+p-1} \text{ is real}),$$

then $f(z)\in S_r^*(\alpha)$ for $0\le r\le r_0$ with

$$\delta=\sqrt{\sum_{n=1}^{\infty}\frac{1}{(n+p-1)^2}}+p\alpha\sqrt{\sum_{n=1}^{\infty}\frac{1}{(n+p-1)^4}}$$

Now, for $p=1$

$$\begin{aligned}\delta&=\sqrt{\sum_{n=1}^{\infty}\frac{1}{n^2}}+\alpha\sqrt{\sum_{n=1}^{\infty}\frac{1}{n^4}}\\&=\sqrt{\zeta(2)}+\alpha\sqrt{\zeta(4)}\\&=\pi\left(\frac{1}{\sqrt{6}}+\frac{\pi\alpha}{3\sqrt{10}}\right)\end{aligned}$$

Further, letting $\alpha=0$, we have that

$$\delta=\sqrt{\frac{\pi^2}{6}}=\sqrt{1.644934}\approx 1.282554$$

and

$$r_0=\sqrt{\frac{\sqrt{6}}{\pi+\sqrt{6}}}\approx 0.6618212$$

Now, for $p=2$ and $\alpha=0$, we have

$$\delta=\sqrt{\sum_{n=1}^{\infty}\frac{1}{(n+1)^2}}=\sqrt{0.644934}\approx 0.8030778$$

and $r_0\approx 0.7447196$

Similarly, for $p=3$ and $\alpha=0$, we have

$$\delta = \sqrt{\sum_{n=1}^{\infty} \frac{1}{(n+2)^2}} = \sqrt{0.394934} \approx 0.6284377$$

and $\quad r_0 \approx 0.7836361$

and so on.

So, we arrive at a result that as p increases r_0 also increases but δ decreases.

Theorem 2.5. *A function* $f(z) \in \Sigma_r(p)$ *belongs to the class* $\mathrm{C}_{\mathrm{r}}(\alpha)$ *for* $0 \le r \le r_1$, *where*

$$r_1 = \sqrt{1 - \frac{\xi}{\xi + 1 - \alpha}}$$

and

$$\xi = \sqrt{\sum_{n=1}^{\infty} (n+p-1)^3 \left|a_{n+p-1}\right|^2} + p\alpha \sqrt{\sum_{n=1}^{\infty} (n+p-1) \left|a_{n+p-1}\right|^2}$$

Example 2: Let us consider the function $f(z)$ given by

$$f(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} \frac{1}{(n+p-1)^2 \sqrt{(n+p-1)}} e^{i\theta_{n+p-1}} z^{n+p-1} \qquad (\theta_{n+p-1} \text{ is real}),$$

then $f(z) \in C_r(\alpha)$ for $0 \le r \le r_1$ with

$$\xi = \pi \left(\frac{1}{\sqrt{6}} + \frac{\pi\alpha}{3\sqrt{10}} \right) \qquad \text{for } p = 1$$

Taking $\alpha = 0$ we obtain

$$\xi = \frac{\pi}{\sqrt{6}} \approx 1.282554$$

and

$$r_1 = \sqrt{\frac{\sqrt{6}}{\pi + \sqrt{6}}} \approx 0.6618212$$

Remarks

1. From Examples 1 and 2 we find that the radius problems for starlikeness and convexity of functions $f(z) \in \Sigma_r(p)$ gives us the result that $r_0 = r_1$.
2. Taking $a_{n+p-1} \geq 0$, $a_0 = 0$ and $r = 1$ in (2.1), Theorem (2.1) leads to the corresponding result obtained by M.L. Mogra in Corollary 2 [2].
3. The results of Theorem (2.1) and (2.2) reduce to the corresponding results obtained in [3] by taking $p = 1, a_0 = 0$ and $r = 1$.
4. There are thirty-nine articles discussing various properties of classes consisting of univalent, starlike, convex, multivalent, and meromorphic functions in the book by Srivastava and Owa [1].

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EXTERNAL LINKS

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